Table 1 Experimental data on hybrid rocket motor

Aluminum content, %	Pyrometer temperature reading, °F	Maximum chamber pressure, psia
18	4880	395
18	4840	370
18	5110	400
18	4730	365
18	4890	300
40	5080	350

of exhaust products. An average specific heat for the exhaust products would be, conservatively, 0.25 (cal/g °C). For 1.0 g of NH_4ClO_4, therefore,

$$\Delta T = (-Q/mCp)$$

where

ΔT = temperature change, C°
Q = quantity of heat in calories
m = mass in grams
C_p = specific heat, cal/g °C

$$\Delta T = (-28.2/1.67 \times 0.25) = -67.5°\text{C}$$
$$\Delta T = -122°\text{F}$$

This temperature change is not considered significant. It is acknowledged, however, that such a mixing of the solid fuel and gaseous oxidizer will never be exactly the same as in the true solid propellant, and good performance will depend on proper oxidizer injector and motor design.

Experimental Studies

Fuel grains were molded and machined in the form of a hollow cylinder, 1.78 in. o.d. × 0.5 in. i.d. × 1.65 in. These were designed to fit within a small water-cooled rocket motor. An injector directed oxygen to the fuel grain. Several binder systems and binder additives were used in the grain formulations. A hydrogen-oxygen pilot was used to ignite the grain. After 1 to 2 sec of pilot ignition, the main oxygen valve was opened. All grains ignited without difficulty, and burned smoothly. The chamber pressure rose rapidly and reached a steady state value, in most cases, before the grains were consumed. Chamber pressures as high as 400 psi were recorded. Firing durations were of the order of 4 sec.

A refractory material was placed downstream of the nozzle exit, and an optical pyrometer was sighted on its surface to obtain an estimate of the combustion temperature. The data from these runs are represented in Table 1. Generally, the data indicate that the combustion temperature was in the vicinity of 5000°F. This temperature is only slightly below the 5300°–6400°F range for ammonium perchlorate-aluminum systems.

Comments

Covariance Matrix Approximation

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The determinant of the covariance matrix of a trivariate normal distribution is a figure of merit which relates to error volumes in tracking and prediction problems. In computing this determinant, neglect of the covariance elements leads to pessimistic estimates of system accuracy, hence is a safe approximation for preliminary analysis. This property is verified, and a comparison of error volume results obtained with and without inclusion of covariance elements is given for different degrees of correlation between errors.

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The purpose of this note is to call attention to a useful approximation in error analyses involving trivariate normal distributions. By way of background, such a distribution in the error coordinates x,y,z, all assumed to be normalized to zero means, has as density function

$$p(x,y,z) = (2\pi)^{-3/2}|M|^{-1/2} \exp\{-\tfrac{1}{2}[xyz]M^{-1}[xyz]^T\} \quad (1)$$

where -1 and T indicate matrix inverse and transpose. M is the symmetric covariance matrix of the distribution, defined by

$$M = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (2)$$

where σ_{xx} is the variance of x error, $\sigma_{xy} \equiv \sigma_{yx}$ is the covariance of x and y errors, etc. Taking the errors x, y, z to be referred to rectangular coordinate axes, a rotation of these axes can always be accomplished by an orthogonal transformation such that, in the new axis system (denoted by primes), the errors x',y',z' are independent. In the primed system, the covariance matrix M' has as elements of its principal diagonal the variances $\sigma_{x'x'},\sigma_{y'y'},\sigma_{z'z'}$ of x',y',z' errors, with all off-diagonal elements being zero. The determinant $|M'|$ has the property that $4/3\pi|M'|^{1/2}$ is equal to the volume of an ellipsoid aligned with the primed axes and having semi-axes $\sigma_{x'} \equiv (\sigma_{x'x'})^{1/2}$, $\sigma_{y'},\sigma_{z'}$. Corresponding to this error volume is a probability level of about $P = 0.2$, and other levels can be associated with similar ellipsoids defined by semi-axes $k\sigma_{x'},k\sigma_{y'},k\sigma_{z'}$, where k is any positive constant.

In many vehicle tracking and prediction problems where x, y, z are rectangular position coordinate errors, the sizes of the preceding error volumes (for different k) are taken as measures of system accuracy, since it is within such volumes, centered about the measured or computed position of the vehicle, that it may be expected to lie with different probability levels P. Thus, $|M'|$ is a fundamental figure of merit. To calculate $|M'|$, it is not necessary to transform the original distribution to the primed system in which the errors are independent, because from the orthogonality of the required transformation it follows that $|M| = |M'|$. That is, it is sufficient to work with the determinant of the covariance matrix of the original distribution. Even allowing for this property, however, it is still inconvenient in preliminary analyses to have to deal with the covariance elements. The question then naturally arises as to the type and size of error introduced by approximating $|M|$ as merely the product of the principal diagonal variance elements only, i.e., by assuming all covariances to be zero.

The point of the present note is simply this: the foregoing approximation is a pessimistic or "safe" one, in that the determinant so obtained is always greater than (or equal to) the true value of $|M|$ with covariance elements included. Relating $|M|$ to error volumes, if the accuracy requirements for a particular system are satisfied for $|M|$ approximated in this way, a more thorough analysis incorporating covariance is obviated, for the results can only be more (or equally) favorable in terms of system performance exceeding requirements.

To prove the inequality $\sigma_{xx}\sigma_{yy}\sigma_{zz} \geq |M|$, introduce correlation coefficients $\rho_{xy} \equiv \sigma_{xy}/(\sigma_{xx}\sigma_{yy})^{1/2}$ and ρ_{xz}, ρ_{yz}, similarly defined. Then, by expansion of $|M|$,

$$|M| = \sigma_{xx}\sigma_{yy}\sigma_{zz}\,[1 - (\rho_{xy}^2 + \rho_{xz}^2 + \rho_{yz}^2 - 2\rho_{xy}\rho_{xz}\rho_{yz})] \quad (3)$$

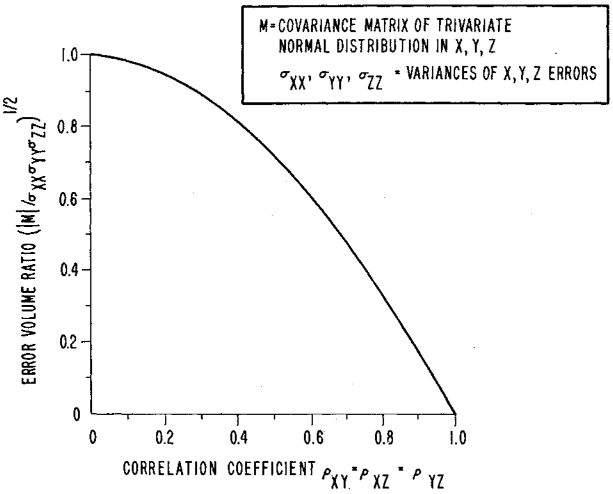

Fig. 1 Effect of correlation on error volume approximation.

In order for the density function (1) to be defined, $|M|>0$; hence it remains only to show that the expression involving the ρ's in parentheses in (3) is nonnegative. This follows from the inequalities

$$\rho_{xy}^2 + \rho_{xz}^2 + \rho_{yz}^2 \geq \rho_{xy}^2 + \rho_{xz}^2 \geq |2\rho_{xy}\rho_{xz}| \geq |2\rho_{xy}\rho_{xz}\rho_{yz}| \quad (4)$$

where use is made of the property that any correlation coefficient, by definition, cannot exceed unity in absolute value. Combining the first and last sections of (4),

$$\rho_{xy}^2 + \rho_{xz}^2 + \rho_{yz}^2 - 2\rho_{xy}\rho_{xz}\rho_{yz} \geq 0 \quad (5)$$

and, since $|M| > 0$, it is concluded from (3) that

$$\sigma_{xx}\sigma_{yy}\sigma_{zz} \geq |M| \quad (6)$$

To give an idea of how the degree of correlation between the three error variables affects the error volumes previously described, the ratio $|M|^{1/2}/(\sigma_{xx}\sigma_{yy}\sigma_{zz})^{1/2}$ obtained from (3) is plotted in Fig. 1 as a function of (positive) correlation coefficient. This ratio is equivalently the ratio of error volumes calculated with and without the covariance elements included in M. For simplicity, the three paired correlation coefficients ρ_{xy}, ρ_{xz}, ρ_{yz} are taken here as equal, but the effect of unequal coefficients can be roughly inferred from these results. Figure 1 indicates that a relatively high degree of correlation must exist between the error variables before the error volume is significantly altered from its approximate value obtained by considering the errors to be independent. For example, a common correlation coefficient of 0.5 reduces the actual error volume to about 0.7 of its approximate value.

Use of the Adjoint System in the Solution of Two-Point Boundary Value Problems

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Some papers[1, 2] have appeared recently on the use of the adjoint system in solving a two-point boundary value problem for a system of n first order differential equations. While the method described is workable, it does not seem to have any pronounced advantage over a more obvious method which makes use of the equations of variation directly and thereby avoids some of the rather confusing aspects of the adjoint equations. The special two-point boundary value problem considered is that of solving the equations

$$\dot{x}^i = f^i(x^1, \ldots, x^n, t) \qquad (i = 1, \ldots, n) \quad (1)$$

subject to the initial conditions

$$x^i(t_0) = a^i \qquad (i = 1, \ldots, r) \quad (2)$$

and the final ($t_1 > t_0$) conditions

$$x^i(t_1) = b^i \qquad (i = r + 1, \ldots, n) \quad (3)$$

Let $y^i(t)$ be the solution (a set of functions) of (1) subject to the initial conditions

$$y^i(t_0) = a^i \qquad (i = 1, \ldots, n)$$

where $a^{r+1}, \ldots, a^n$ are estimates of the unspecified initial conditions which will produce the final conditions (3), and suppose that the final values are found actually to be

$$y^i(t_1) = c^i \qquad (i = r + 1, \ldots, n)$$

If

$$x^i(t) = y^i(t) + \xi^i(t) \quad (4)$$

the equations of variation are

$$\dot{\xi}^i = \sum_{j=1}^{n} f_j^i \xi^j \quad (5)$$

where f_j^i denotes the partial derivative of $f^i(x^1, \ldots, x^n, t)$ with respect to x^j after the solution functions $y^i(t)$ have been substituted for the x^i, so that the f_j^i are known functions of t. The initial and final conditions to be imposed on the solution of (5) are found from (4) to be

$$\xi^i(t_0) = 0 \qquad (i = 1, \ldots, r) \quad (6)$$

and

$$\xi^i(t_1) = b^i - c^i = \beta^i \qquad (i = r + 1, \ldots, n) \quad (7)$$

where the β^i are known. The problem of solving (5) subject to (6) and (7) has the same character as that of solving (1) subject to (2) and (3), but the linearity of Eq. (5) resulting from the neglect of higher order terms can be exploited to obtain a solution. Once the solution has been obtained, the values for $i = r + 1, \ldots, n$ of $\xi^i(t_0)$ can be found and the unspecified initial values for $i = r + 1, \ldots, n$ of $x^i(t_0)$ are approximately $a^i + \xi^i(t_0)$. A new solution y^i is then obtained with these estimates, and the entire process is repeated until convergence is obtained.

Instead of solving Eqs. (5–7) directly, the method of adjoints introduces a new set of variables λ_i satisfying the adjoint equations

$$\dot{\lambda}_i = -\sum_{j=1}^{n} f_i^j \lambda_j \quad (8)$$

and it is easily seen that any pair of solutions of (5) and (8) satisfies the relation

$$\sum_{i=1}^{n} \xi^i(t_0)\lambda_i(t_0) = \sum_{i=1}^{n} \xi^i(t_1)\lambda_i(t_1) \quad (9)$$

Now consider $n - r$ solutions $\lambda_i^k(t)$ for $k = r + 1, \ldots, n$ of the adjoint equations subject to the final conditions

$$\lambda_i^k(t_1) = 0 \qquad (i = 1, \ldots, r)$$
$$\lambda_i^k(t_1) = \delta_i^k \qquad (i = r + 1, \ldots, n)$$

which define the values $\lambda_i^k(t_0)$. One can now write $n - r$ different versions of Eq. (9)

$$\sum_{i=1}^{n} \xi^i(t_0)\lambda_i^k(t_0) = \xi^k(t_1) = \beta^k \qquad (k = r + 1, \ldots, n) \quad (10)$$

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